Driven diffusive system: A study on large n limit

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Abstract. We study the generalized n component model of a driven diffusive system with annealed random drive in the large n limit. This non-equilibrium model also describes conserved order parameter dynamics of an equilibrium model of ferromagnets with dipolar interaction. In this limit, at zero temperature a saddle point approximation becomes exact. The length scale in the direction transverse to the driving field acquires an additional logarithmic correction in this limit.

PACS. 64.60.Ht Dynamic critical phenomena – 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.)

1 Introduction

In nature, a large class of phenomena are far-from-equilibrium in origin. Unlike the equilibrium systems, depending on the complexity and the relevant questions involved, there are different ways to treat non-equilibrium systems. In equilibrium, due to the random energy imparted by the environment (thermal noise), all possible energy states are accessible and there is no net flow of probability (*i.e.* no net current) from one configuration to the other. As a result there is a straight forward way of obtaining the free energy and hence the thermodynamic properties by just summing over all the Boltzmann factors $\exp[-E/K_{\mathrm{B}}T]$ for all the states, where E is the energy of the state. In general, although one may write down a master equation describing the time evolution of the probability of a system being in a state, in the large time limit, the equilibrium dynamics ensures a Boltzmann distribution. This behavior is preserved by the so called detailed balance principle. This shows up also in the Langevin equation which describes the time evolution of the degrees of freedom (say *e.g.* the order parameter in a spin system, or the height of a growing surface). In the equilibrium case, the Langevin equation can be obtained from the Hamiltonian representing all the relevant interactions of the system. However, in the non-equilibrium cases, such a description through a Hamiltonian is not possible due to one or more terms. These terms are purely of non-equilibrium origin and appear due to, say, a finite driving force in the driven dynamics of particles [1] or in random deposition of particles in the case of crystal growth [2]. There are non-equilibrium systems where detailed balance may also hold good. In these cases the dynamics reduces

to the equilibrium dynamics of a system described with appropriate Hamiltonian [2].

These aspects of non-equilibrium dynamics are present in different variants of driven diffusive systems (DDS) [1] where particles with diffusive dynamics have biased hopping due to a driving force in a particular direction. In the case of fixed time independent driving force, a formulation of the Langevin equation through a Hamiltonian is not possible due to the term responsible for the drive. This is not necessarily true for all the driven systems. In particular, we shall later be concerned with a driven system with an annealed random drive where the problem effectively reduces to the equilibrium dynamics of a specific magnetic system. In general, without the driving force, the equilibrium system is like a lattice gas where particles randomly occupy lattice sites and also due to a nearest neighbor attraction a particle prefers to have its next neighboring site occupied by another particle. In this sense the equilibrium static properties are the same as that of the Ising model with a ferromagnetic interaction. Even though, the dynamics, unlike the spin case, is different due to the total particle number conservation in the lattice gas, the static properties of the two models are the same due to the equivalence of canonical and grand-canonical ensembles. With a finite driving force [3], the dynamics of lattice gas is significantly different from the equilibrium order parameter conserving dynamics. Three key features, such as, the boundary condition, particle number conservation and the driving force together affect the dynamics in a non-trivial way and as a result one observes generic power law correlation far above the criticality and anisotropy in the exponents of different correlation functions [1].

Due to the time dependent probability distribution, and lack of detailed balance, many questions remain unresolved for this system. Although in the steady state, where

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the distribution settles to a time independent form, one can write down a local detailed balance form for the transition rates in the Master equation, obtaining the physical properties from this is, in practice, difficult. However, even working with very small lattices, certain interesting features can be observed [4,5]. For example, on a small size half filled lattice, in the ground state, unlike the equilibrium case, one can see a strip-like occupied region in the presence of a large driving force. A careful analysis of the internal energy fluctuation and specific heat also reveals new consequences of the driving force [1].

In view of the complications, it is, therefore, often useful to adopt different approximation schemes or numerical techniques. Since in many cases the interest is mostly on long wavelength and long time features, the continuum formulation in terms of the Langevin equation comes quite handy. The continuum formulation provides a fruitful basis for performing the mean field type calculations or field theoretic renormalization group analysis [6] with the latter being especially successful in explaining the new features at the critical point. In this scheme, the existence of a non-equilibrium stable fixed point that governs the new universal properties, anisotropic non-Gaussian scaling exponents associated with it, is clearly observed. The fact that far above the criticality, the two point or three point correlation exhibits a power law behavior, can be directly shown using the above continuum formulation and averaging with $\exp[-\mathcal{J}]$ with $\mathcal{J}$ as the dynamic functional. This weight factor essentially plays a role that is analogous to the Boltzmann weight for the equilibrium system.

Although significant progress has been made in understanding the properties far above and at criticality, the situation below criticality is relatively unclear. Below the critical temperature, one would expect phase separated occupied and unoccupied regions provided the filling is not too far away from the half filling where one would remain out of the coexistence regime. In the presence of the driving force, the shape of the coexistence curve changes from the equilibrium situation [7] and also the shape of the phase separated regions gets distorted. The basic questions that still remain are related to these aspects. Although numerical simulations show that the particle occupied regions are somewhat elongated in the direction of the field, sufficient analytical progress is yet to be made.

Along with this development, several variants of this simple case of driven lattice gas have appeared in order to understand more realistic phenomena in nature. One simple choice of the boundary condition is the periodic one in both transverse and longitudinal direction. In that case the system looks like a torus with the driving field, possibly a uniform electric field looping around it. To obtain such a situation in laboratory, one would require a magnetic flux increasing linearly in time. Therefore due to these practical purposes, a demand for more abundant situations with non-uniform drive or different boundary conditions was obvious. Thus came up a simpler but more realistic model of DDS with annealed random drive [8,9]. Interestingly, this model exhibits new universal properties that are different from the uniformly driven Ising model.

In this paper, we are interested in studying a driven diffusive system with annealed random drive with n component order parameter in the large n limit. In this limit the problem is exactly solvable at zero temperature. It is known that the large time behavior of a system, quenched below the critical temperature, is characterized by the equal time structure factor

$$c(k,t) = \langle \phi_i(\mathbf{k},t)\phi_i(-\mathbf{k},t)\rangle = L^\alpha(t)F(kL(t)), \quad (1)$$

where $L(t) \sim t^{1/z}$, with z being the dynamic exponent, represents the characteristic size of the correlated regions growing with time [10]. In the above definition, the subscript i denotes the component of the order parameter field ϕ. This scaling law is preserved in the case of quenching of a system with a non-conserved order parameter. However, this scaling form is not maintained in a system with a conserved order parameter. In the latter case one has two length scales, obeying two different scalings [11,12]. While the growth of one length scale with time is governed by the usual dynamic exponent, for the other length scale, an additional time dependence appears. The reason for such multi-scaling is understood to be the non-commutativity of the large n and large t limits [13].

2 Model

In the lattice gas systems with a fixed total number of particles, there is an underlying continuity equation obeyed by the particle density ρ, $\partial_t\rho = -\nabla J$, where J is the current. In the absence of the driving force, the current can be expressed as the gradient of the chemical potential $-\lambda\nabla\mu$, where λ is the transport coefficient, $\mu = \delta\mathcal{H}/\delta\rho$, with $\mathcal{H}$ as the appropriate lattice gas Hamiltonian. With an attractive interaction among the particles, the complete description of the dynamics involves only one field, the particle density which plays the role of the order parameter. Near equilibrium, the equation of motion for the conserved dynamics of such attracting particles can be written using the continuity equation and the Hamiltonian, expressed in terms of the local magnetization $\phi(\mathbf{x},t) = 2\rho(\mathbf{x},t) - 1$, as

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \{\frac{1}{2}(\nabla\phi)^2 + \frac{\tau}{2}\phi^2 + \frac{u}{4!}\phi^4\}. \quad (2)$$

The drive, which is actually responsible for the far-from-equilibrium behavior, gives rise to an additional current, which must vanish if there is no hole or no particle. Thus, the simplest choice for the current may be $J_E = 4\rho(1-\rho)E = (1-\phi^2)E$, where E is the strength of the driving field. The system is anisotropic due to the driving force in a particular direction, henceforth denoted by z. In the equation of motion, this z direction and the transverse $d-1$ dimensional space are distinguished by different coefficients [1] as shown below

$$\partial_t\phi(\mathbf{x},t) = \lambda\{(\tau_\perp - \nabla^2)\nabla^2\phi + (\tau_\parallel - \alpha_\parallel\partial^2)\partial^2\phi$$
$$- \alpha_\times\partial^2\nabla^2\phi + \frac{u}{3!}(\nabla^2\phi^3 + \chi\partial^2\phi^3) + E\partial\phi^2\}$$
$$- (\nabla\xi + \partial\zeta). \quad (3)$$

Here and in the following discussion, ∇ in general extends over the $d-1$ dimensional space and ∂ denotes the derivative with respect to z only. The last two terms in the Langevin equation originate from the usual noisy part of the current and have short range correlations both in space and time with variances $n_\perp$ and $n_\parallel$. The coefficients associated with the derivatives with respect to transverse or parallel coordinates have subscripts $\parallel$ and $\perp$ respectively. As the critical temperature is approached, the most realistic choice of parameters is $\tau_\parallel > 0$ and $\tau_\perp \to 0$ [14]. In fact, it can be explicitly shown that while the transverse direction, and hence $\tau_\perp$, remain unaffected by the driving force, $\tau_\parallel$ is renormalized upwards due to the driving force and remains uniform for the whole system.

The focus of the present paper is on an annealed random drive which has a correlation $\langle E(\mathbf{x},t)E(\mathbf{x}',t')\rangle = \sigma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. A well-known starting point for the dynamics is to write down the dynamic functional $\mathcal{J}$ that involves the Martin-Siggia-Rose response field $\tilde{\phi}$ [15]. After averaging out the drive, and ignoring all the redundant or irrelevant terms, the generating functional can be written as

$$\mathcal{J}_b = \int \mathrm{d}t\, \mathrm{d}^d x\, \lambda \left\{ \tilde{\phi}(\mathbf{x},t)[\lambda^{-1}\partial_t - (\tau - \nabla^2)\nabla^2 - \partial^2]\phi(\mathbf{x},t) \right. $$
$$\left. -\frac{u}{3!}\tilde{\phi}\nabla^2\phi^3 + \tilde{\phi}\nabla^2\tilde{\phi} \right\}. \qquad (4)$$

Since the non-Gaussian interaction term has no longitudinal operator, it does not affect $\tau_\parallel$, which, as a result, has been considered to be a constant (unity) in (4). The coefficient of the last term in the curly bracket has also been set to unity by suitable reparametrization. For convenience we have denoted $\tau_\perp$ in this equation by τ.

It is now appropriate to point out that there exists a direct connection between the driven system and uniaxial ferromagnets with dipolar interaction [8,16,17]. The generating functional can be written in the detailed balance form as

$$\mathcal{J}_b = \int \{\tilde{\phi}\partial_t\phi - \tilde{\phi}\lambda\nabla^2[\frac{\delta\mathcal{H}_d}{\delta\phi} - \tilde{\phi}]\}, \qquad (5)$$

where

$$\mathcal{H}_d[\phi] = \int_k \frac{1}{2}\phi(-k)S_0^{-1}(k)\phi(k)$$
$$+ \frac{u}{4!}\int_{k_1,k_2,k_3}\phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1-k_2-k_3), \qquad (6)$$

with $S_0^{-1}(k) = k_\perp^{-2}[\tau k_\perp^2 + k_\perp^4 + k_\parallel^2]$. This Hamiltonian represents the static system of uniaxial ferromagnet with dipolar interaction [16,17]. Therefore studying this driven system with annealed random drive also implies studying the order parameter conserved dynamics of uniaxial ferromagnets.

3 Results

Our starting point for the large n analysis is essentially an n component generalization of the Hamiltonian $\mathcal{H}_d$ appearing in the detailed balance form above. To proceed

further, we start with the Langevin equation which represents the dynamics of the driven diffusive system of our interest as well as the conserved dynamics of the uniaxial dipolar system

$$\frac{\partial\phi_i}{\partial t} = \nabla_\alpha[\nabla_\alpha\frac{\partial\mathcal{H}}{\partial\phi_i} + \zeta_\alpha(x,t)], \qquad (7)$$

where $\zeta_\alpha(x,t)$ represents the short range correlated Gaussian noise. Here α ($= 1, 2,...d$) stands for the dimension of the space and i ($= 1,....n$) denotes the order parameter component. For the present discussion, we shall restrict ourselves to the zero temperature situation where we need not bother about the noise term.

In Fourier space the Langevin equation becomes

$$\frac{\partial\phi_i}{\partial t} = (-\tau k_\perp^2 - k_\perp^4 - k_\parallel^2)\phi_i - \frac{u_0}{3!}a(t)k_\perp^2\phi_i, \qquad (8)$$

where $\phi^2 = \sum_{j=1}^n \phi_j^2 = n\langle\phi_i^2\rangle = a(t)$. Solving (8), we obtain

$$\phi_i(k,t) = \phi_i(k,0)\mathrm{e}^{-(\tau k_\perp^2 + k_\perp^4 + k_\parallel^2)t - (u_0/3!)k_\perp^2 b(t)}, \qquad (9)$$

with $b(t) = \int_0^t a(t')\mathrm{d}t'$. Therefore

$$\langle\phi_i(x,t)\phi_i(x,t)\rangle = \int \frac{\mathrm{d}k}{(2\pi)^d}\frac{\Delta}{n}$$
$$\times \mathrm{e}^{-(2\tau k_\perp^2 + 2k_\perp^4 + 2k_\parallel^2)t - (u_0/3)k_\perp^2 b(t)}, \qquad (10)$$

where we have used the initial condition

$$\langle\phi_i(\mathbf{k},0)\phi_j(\mathbf{k}',0)\rangle = \frac{\Delta}{n}\delta_{ij}\delta(\mathbf{k}+\mathbf{k}'). \qquad (11)$$

Therefore we have

$$a(t) = \frac{\Delta}{(2\pi)^d}\int \mathrm{d}^{d-1}k_\perp \mathrm{d}k_\parallel \mathrm{e}^{-(2\tau k_\perp^2 + 2k_\perp^4)t - 2k_\parallel^2 t - (u_0/3)k_\perp^2 b(t)}$$
$$= \frac{\Delta}{(2\pi)^d}(\frac{\pi}{2t})^{1/2}\frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}\int \mathrm{d}k_\perp k_\perp^{d-2}\mathrm{e}^{-(2\tau' k_\perp^2 + 2k_\perp^4)t}, \qquad (12)$$

where $\tau' = \tau + \frac{u_0}{6}\frac{b(t)}{t}$. Substituting $(\frac{1}{\tau'})^{1/2}k_\perp = x$, this equation simplifies to

$$a(t) = \frac{\Delta}{(2\pi)^d}(\frac{\pi}{2t})^{1/2}\frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}(\tau + \frac{u}{6}\frac{b(t)}{t})^{(d-1)/2}$$
$$\times \int \mathrm{d}x\, x^{d-2}\mathrm{e}^{-(x^2+x^4)2t\tau'^2}. \qquad (13)$$

To solve this equation, we shall consider different dimensions separately. At $d = 2$, the saddle point is at the origin. Therefore we obtain

$$a(t)^2 = B_0(\tau + \frac{u}{6}\frac{b(t)}{t})/t, \qquad (14)$$

where $B_0 = \Delta^2/4\pi$. Differentiating this equation once, we have

$$2t^2 a(t)\frac{\mathrm{d}a(t)}{\mathrm{d}t} = \frac{uB_0}{6}a(t) + B_0\tau - 2ta(t)^2. \qquad (15)$$

The numerical solution of this equation in terms of three constants c_1, c_2 and c_3 is

$$a(t) = c_1 + \frac{c_2}{t^{1/2}} + c_3 \frac{\log t}{t}. \tag{16}$$

In the large time limit, the dominant time dependence is of the form $a(t) \sim t^{-1/2}$. Finally, since the factor $2t\tau'^2$ in the argument of the exponential approaches infinity as $t \to \infty$, the saddle point approximation turns out to be an exact one.

Let us consider $d = 3$. Starting with (13), we again replace the integral by the maximum value of the integral. This occurs at

$$x^2 = \frac{-2c_0 \pm \sqrt{4c_0^2 + 16c_0}}{8c_0}, \tag{17}$$

where $c_0 = 2\tau'^2 t$. As $t \to \infty$, one can approximate $x^2 \sim 1/2c_0$. Therefore in the large time limit,

$$a(t) = \frac{\Delta}{4(2\pi)^{3/2}} \frac{1}{t} \exp[-(1/2 + 1/4c_0)]. \tag{18}$$

Therefore, at $d = 3$, $a(t)$ scales as $1/t$ at large time. To obtain the relevant length scale, we rewrite $\phi_i(k,t)$ as

$$\phi_i(k,t) = \phi_i(k,0)e^{-k_\parallel^2 t}e^{[(\tau t + (u/3!)b(t))^2/4t]\{1-((k_\perp/k_L)^2+1)^2\}}. \tag{19}$$

Apart from the length scale in the parallel direction growing as $L_\parallel \sim t^{1/2}$, we find that the length scale in the direction perpendicular to the driving force acquires a logarithmic correction in addition to the expected scaling behavior. At $d = 3$, the length scale in the transverse direction scales with time as

$$L_\perp \sim \frac{t^{1/4}}{(1 + u \ln t/6t^{1/2})^{1/2}} \simeq t^{1/4} \text{ for } t \to \infty. \tag{20}$$

This is somewhat different from the multiscaling observed before, where there are two different scaling lengths. Here due to the anisotropy originally present in the system, the transverse and longitudinal length scales $L_\perp$ and $L_\parallel$ respectively, have different scaling properties with time even at the Gaussian level. $L_\perp$, due to the non-Gaussian interaction, relevant only in the transverse direction, has modified scaling behavior due to a logarithmic correction in addition to the scaling expected at the Gaussian level.

As expected, the scaling of $L_\parallel$ remains unaltered from the Gaussian level.

To summarize, we have studied the large n limit of a generalized n component driven diffusive model with annealed randomness at zero temperature. This model also corresponds to the order parameter conserved dynamics of uniaxial ferromagnets with dipolar interaction. This model is exactly soluble in the large n limit. The scaling of the length scale transverse to the drive is only modified due to an additional logarithmic correction. This is quite different from the multiscaling behavior observed before in conserved dynamical models.

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